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Letters to the Editor

A barometric formula in relativistic magnetohydrodynamics

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Abstract. It is shown that if the space–time of a system composed of a perfect fluid coupled to a frozen-in magnetic field has two Killing vector fields, one of them collinear to four-velocity and the other proportional to the magnetic field, a barometric formula is obtained from which the influence of the magnetic field on the pressure distribution is found.

Recently Dehnen and Obregon (1971) have obtained a general relativistic barometric formula for an incompressible fluid without internal energy. Theoretical studies of the gravitational collapse problem show that in any real astrophysical case it is necessary to take into account the internal energy and the effects of magnetic fields.

In this work we shall obtain, in the framework of general relativity, a barometric formula for a system composed of a perfect fluid coupled to a frozen-in magnetic field (the case of magnetohydrodynamics).

Let u^{μ} be the unitary four-velocity of a fluid from a Riemann space of signature $+, -, -, -$. We suppose that there exists a time-like Killing vector field ξ_{μ} collinear to u_{μ}

$$\xi_{\mu;\nu}+\xi_{\nu;\mu} = 0 \qquad \xi^2 = \xi^{\mu}\xi_{\mu} > 0 \qquad u_{\mu} = \frac{\xi_{\mu}}{\xi}. \tag{1}$$

In this case the acceleration vector $\dot{u}_{\nu}$, the expansion scalar θ and the shear tensor $\sigma_{\mu\nu}$ are

$$\dot{u}_{\mu} = u_{\mu;\nu}u^{\nu} = -(\ln \xi)_{,\mu} \tag{2}$$

$$\theta = u^{\alpha}{}_{;\alpha} = 0 \tag{3}$$

$$\sigma_{\mu\nu} = u_{(\mu;\nu)} - u_{(\mu}\dot{u}_{\nu)} - \tfrac{1}{3}\theta(g_{\mu\nu}-u_{\mu}u_{\nu}) = 0. \tag{4}$$

Thus the motion of such a fluid satisfies the Born conditions for rigid body motion (Anderson 1967).

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It is easy to show that we have to deal with a stationary gravitational field if we consider a comoving frame of reference and the line element of the form

$$ds^2 = dx^{0^2} + 2g_{0i}\, dx^0\, dx^i + g_{ik}\, dx^i\, dx^k. \tag{5}$$

The main equations of relativistic magnetohydrodynamics are the equations of conservation

$$T_\beta{}^\alpha{}_{;\alpha} = 0 \tag{6}$$

and the Maxwell equations

$$(u^\alpha h^\beta - u^\beta h^\alpha)_{;\alpha} = 0 \tag{7}$$

where $T_{\alpha\beta}$ is the energy tensor:

$$T_{\alpha\beta} = \left(\rho + \frac{\rho\epsilon}{c^2} + \frac{p}{c^2}\right) u_\alpha u_\beta - \frac{p g_{\alpha\beta}}{c^2} - \mu\left\{\left(\frac{1}{2} g_{\alpha\beta} - u_\alpha u_\beta\right)|\boldsymbol{h}|^2 + h_\alpha h_\beta\right\}. \tag{8}$$

The symbols used here have their usual significance (Lichnerowicz 1967).

The projection of equation (6) on the local time-axis u^β according to (3) gives

$$\left(\rho + \frac{\rho\epsilon}{c^2} + \frac{p}{c^2} + \mu|\boldsymbol{h}|^2\right)_{,\alpha} u^\alpha - \left(\frac{p}{c^2} + \frac{1}{2}\mu|\boldsymbol{h}|^2\right)_{,\beta} u^\beta - \mu h^\alpha h_{\beta;\alpha} u^\beta = 0. \tag{9}$$

Then we may use the relations

$$u^\alpha h_\alpha = 0 \qquad \xi^\alpha h_\alpha = 0$$

in order to obtain

$$h^\alpha h_{\beta;\alpha} u^\beta = 0 \tag{10}$$

and (9) becomes

$$\left(\rho + \frac{\rho\epsilon}{c^2} + \frac{1}{2}\mu|\boldsymbol{h}|^2\right)_{,\alpha} u^\alpha = 0. \tag{11}$$

Using (3) and (10), from the Maxwell equations (7) it follows that:

$$|\boldsymbol{h}|^2{}_{,\alpha} u^\alpha = 0 \tag{12}$$

that is, $|\boldsymbol{h}|^2$ is constant along the stream lines.

If ρ is conservative in the motion of the fluid (locally adiabatic motion)

$$\rho_{,\alpha} u^\alpha = 0 \tag{13}$$

from (11) and (12) we obtain

$$\epsilon_{,\alpha} u^\alpha = 0. \tag{14}$$

As the specific internal energy can be considered as a given function of two thermodynamical variables of the fluid, ρ and p for instance, from (13) and (14) it follows that the pressure is constant with respect to time, that is

$$p_{,\alpha} u^\alpha = 0. \tag{15}$$

According to (2) and (12)–(15), the equations of conservation (6) become

$$(\rho + \rho\epsilon/c^2 + p/c^2 + \mu|\boldsymbol{h}|^2)(\ln \xi)_{,\beta} + (p/c^2 + \tfrac{1}{2}\mu|\boldsymbol{h}|^2)_{,\beta} + \mu h^\alpha{}_{;\alpha} h_\beta + \mu h^\alpha h_{\beta;\alpha} = 0. \tag{16}$$

In the comoving frame, with the metric (5) we obtain

$$h^{\alpha}{}_{;\alpha} = 0. \tag{17}$$

We assume further the existence of a space-like Killing vector field proportional to h_{μ}. Then if ρ, ϵ and $|\boldsymbol{h}|^2$ are constant we can integrate equation (16) and obtain the barometric formula

$$p = (\rho c^2 + \rho\epsilon + \mu c^2|\boldsymbol{h}|^2)\left(\frac{\xi_s}{\xi} - 1\right)$$

where ξ_s is the absolute value of the Killing vector at the surface of the considered domain of space–time.

We note that the magnetic field contributes to the fluid pressure twice: through the proper energy density $\frac{1}{2}\mu c^2|\boldsymbol{h}|^2$ and through the magnetic pressure $\frac{1}{2}\mu c^2|\boldsymbol{h}|^2$. This conclusion is in full accordance with the fundamental concepts of relativistic magnetohydrodynamics.

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An exact solution of the multiatom, multimode model Hamiltonian of quantum optics

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Abstract. Exact expressions for the eigenvectors and eigenvalues of a Hamiltonian of great importance in quantum optics are derived.

The Hamiltonian we shall consider is

$$H = \sum_{k=1}^{N} a_k{}^{\dagger} a_k w_k + R_3 w_0 + \sum_{k=1}^{N} (g a_k R_+ + g^* a_k{}^{\dagger} R_-) \tag{1}$$

where $R_\alpha = \Sigma_{i=1} \frac{1}{2}\sigma_{i\alpha}$, the $\sigma_{i\alpha}$ being the Pauli spin matrices for the ith atom, and $a_k{}^{\dagger}$ is the creation operator for the kth mode of the electromagnetic field. This Hamiltonian, which describes a system of N electromagnetic modes interacting with M two